

# Josephson instantons and Josephson monopoles in a non-Abelian Josephson junction

Muneto Nitta<sup>1</sup>

<sup>1</sup> *Department of Physics, and Research and Education Center for Natural Sciences,  
Keio University, Hi-yoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan*

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## Abstract

Non-Abelian Josephson junction is a junction of non-Abelian color superconductors sandwiching an insulator, or a non-Abelian domain wall if flexible, whose low-energy dynamics is described by a  $U(N)$  principal chiral model with the conventional pion mass. A non-Abelian Josephson vortex is a non-Abelian vortex (color magnetic flux tube) residing inside the junction, that is described as a non-Abelian sine-Gordon soliton. In this paper, we propose Josephson instantons and Josephson monopoles, that is, Yang-Mills instantons and monopoles inside a non-Abelian Josephson junction, respectively, and show that they are described as  $SU(N)$  Skyrmons and  $U(1)^{N-1}$  vortices in the  $U(N)$  principal chiral model without and with a twisted mass term, respectively. Instantons with a twisted boundary condition are reduced (or T-dual) to monopoles, implying that  $\mathbb{C}P^{N-1}$  lumps are T-dual to  $\mathbb{C}P^{N-1}$  kinks inside a vortex. Here we find  $SU(N)$  Skyrmons are T-dual to  $U(1)^{N-1}$  vortices inside a wall. Our configurations suggest a yet another duality between  $\mathbb{C}P^{N-1}$  lumps and  $SU(N)$  Skyrmons as well as that between  $\mathbb{C}P^{N-1}$  kinks and  $U(1)^{N-1}$  vortices, viewed from different host solitons. They also suggest a duality between fractional instantons and bions in the  $\mathbb{C}P^{N-1}$  model and those in the  $SU(N)$  principal chiral model.

## I. INTRODUCTION

Yang-Mills instantons and magnetic monopoles are two topological solitons studied very well in both physics and mathematics [1]: they are both integrable, admit hyper-Kähler moduli spaces, and their solutions are available through by the Atiyah-Drinfeld-Hitchin-Mannin [2] and Nahm [3] constructions, respectively. They are related by a so-called Nahm transformation that can be now understood as a T-duality in D-brane realization of these objects in type-II string theory [4]. A new twist on these objects found recently was their realizations in the Higgs phase in which gauge symmetry is completely broken when gauge fields are coupled with several Higgs scalar fields in the fundamental representation. In the Higgs phase, there can exist a non-Abelian vortex that has  $\mathbb{C}P^{N-1}$  moduli [5–7] and a non-Abelian domain wall carrying  $U(N)$  moduli [8–10]. Although instantons cannot exist stably in the Higgs phase,  $SU(N)$  instantons can stably exist as  $\mathbb{C}P^{N-1}$  lumps (or instantons) [11] inside a non-Abelian vortex [12, 13] and as  $SU(N)$  Skyrmons inside a non-Abelian domain wall [9]. (The latter setting physically realizes the Atiyah-Manton construction of Skyrmons from instanton holonomy [14].) On the other hand, monopoles are confined by magnetic fluxes in the Higgs phase, and they become  $\mathbb{C}P^{N-1}$  kinks [15–18] inside a non-Abelian vortex [19–22]. Instantons with a twisted boundary condition are reduced (or T-dual) to monopoles, that is known as the Scherck-Schwartz (or twisted) dimensional reduction [23]. This implies inside a vortex in the Higgs phase that  $\mathbb{C}P^{N-1}$  lumps (instantons) with a twisted boundary condition are reduced (T-dual) to  $\mathbb{C}P^{N-1}$  kinks [12]. See Refs. [24–27] as a review of these composite topological solitons. It is, however, not known thus far what it becomes if a monopole resides inside a non-Abelian domain wall.

We further pursue relations of among these topological solitons to find a complete circle. A key ingredient is a recently proposed non-Abelian Josephson junction [28], that is a junction of non-Abelian color superconductors sandwiching an insulator, or a non-Abelian domain wall if it is flexible. As for color superconductors, one can consider either those in dense quark matter at high baryon density [29, 30], or those in supersymmetric gauge theories in the Higgs phase [24–27]. The low-energy dynamics of the non-Abelian Josephson junction can be described by the  $U(N)$  principal chiral model [9], in which the Josephson term in the bulk induces a pion mass term [28]. When a non-Abelian vortex (or color magnetic flux tube) exists in the bulk color superconductor, it is absorbed into the junction if it exists.

The non-Abelian vortex residing inside the junction is referred as a non-Abelian Josephson vortex (or fluxon) [28], that can be described as a non-Abelian sine-Gordon soliton [31] in the  $U(N)$  principal chiral model with the pion mass term. This is a non-Abelian extension of a Josephson vortex described by the usual sine-Gordon soliton [32, 33] in a Josephson junction of metallic superconductors [34]. This correspondence was generalized to higher dimensional Skyrmons [35] and to Yang-Mills instantons [36, 37].

In this paper, we propose Josephson instantons and Josephson monopoles, that is, Yang-Mills instantons and monopoles inside a non-Abelian Josephson junction, respectively, and clarify their relations. We first construct Josephson instantons and monopoles residing a non-Abelian Josephson vortex inside the non-Abelian Josephson junction. If we remove the junction by taking massless limit of the Higgs fields, the configurations go to instantons and monopoles inside the non-Abelian vortex, that is, instanton-vortex [12, 13] and monopole-vortex [19–22] composites known before. Instead, if we remove the vortex in the limit of the vanishing Josephson coupling, there remain bare (unconfined) instantons and monopoles inside the junction. While an instanton becomes a Skyrmion in the junction [9], here we find that a monopole becomes a  $U(1)^{N-1}$  vortex in the  $U(N)$  principal chiral model with the twisted mass inside the junction, for which the monopole charges  $\pi_2[SU(N)/U(1)^{N-1}] \simeq \mathbb{Z}^{N-1}$  coincide with the vortex charges  $\pi_1[U(1)^{N-1}] \simeq \mathbb{Z}^{N-1}$ . We give an explicit ansatz for a single  $U(1)$  vortex for  $N = 2$ . A quite nontrivial check is given by turning on the Josephson interaction in this configuration; We find that there must appear two sine-Gordon solitons with opposite  $\mathbb{C}P^1$  orientations attached to the vortex from its both sides. These sine-Gordon solitons are nothing but Josephson vortices, and so this configuration is precisely the case of a confined monopole. For general  $N$ , we find  $N - 1$  vortices connected or attached by  $N$  sine-Gordon solitons. As mentioned above, a T-duality between monopoles and instantons leads a T-duality between  $\mathbb{C}P^{N-1}$  kinks and  $\mathbb{C}P^{N-1}$  lumps with a twisted boundary condition inside a non-Abelian vortex [12]. Here, we find that  $U(1)^{N-1}$  vortices are dimensionally reduced from (T-dual to)  $SU(N)$  Skyrmons with a twisted boundary condition. The case of  $N = 2$  was found before in Ref. [38], in which numerical solutions were obtained. The  $\mathbb{C}P^{N-1}$  lumps inside a non-Abelian sine-Gordon soliton in the  $U(N)$  principal chiral model are the  $SU(N)$  Skyrmons [39]. Therefore, they are all instantons if we realize the  $U(N)$  principal chiral model inside the junction (non-Abelian domain wall). Thus, our configurations suggest an another duality between the  $\mathbb{C}P^{N-1}$  lumps and the

$SU(N)$  Skyrmions (both instantons in the bulk) as well as that between the  $\mathbb{C}P^{N-1}$  kinks and the  $U(1)^{N-1}$  vortices (both monopoles in the bulk). Since the former (latter) are both instantons (monopoles) viewed from different host solitons, the non-Abelian vortex on one hand and the non-Abelian domain wall on the other hand, this duality may be understood as T-dualities between these host solitons. All these relations are summarized in Fig. 1.

Instantons (or solitons) are fractionalized, that is, a single instanton (soliton) of unit topological charge is decomposed into multiple fractional instantons (solitons) with fractional topological charges in the presence of a twisted boundary condition. Fractional instantons in the  $\mathbb{C}P^{N-1}$  model [12] (see also Refs. [40]) and the Grassmann sigma model [41] in two Euclidean dimensions have been paid renewal interests because their composites with zero instanton charges, called bions, play significant roles in the resurgence of quantum field theory [42–45]. Fractional instantons and bions in the  $O(N)$  model in Euclidean  $N - 1$  dimensions were also studied in Ref. [46], where the cases of  $N = 2$  and 3 correspond to the  $\mathbb{C}P^1$  model in two dimensions and the  $SU(2)$  principal chiral model in three dimensions, respectively. Those in the principal chiral model in three dimensions were studied in Ref. [47]. Our configuration studied in this paper suggests a kind of duality between fractional instantons and bions in these models, and for general  $N$ , those in the  $\mathbb{C}P^{N-1}$  model in two dimensions and  $SU(N)$  principal chiral model in three dimensions.

In a conventional Josephson junction of two metallic superconductors, electrons carry quantum tunneling. Monopoles carry a quantum tunneling in a dual Josephson junction [48], that is a junction of two confinement phases (as dual superconductors) [49], where quarks are confined and monopoles are considered to be condensed. In contrast to this, our case corresponds to unconfined monopoles stably existing inside a usual Josephson junction of two color superconductors, where quarks are condensed and monopoles are confined [50]. Therefore, it suggests that, as dual to this, unconfined quarks can stably exist in a dual Josephson junction of two confinement phases.

This paper is organized as follows. After our model is given in Sec. II, we summarize the non-Abelian Josephson junction and non-Abelian Josephson vortices in Sec. III. In Secs. IV and V, we construct Josephson instantons and monopoles, respectively. Section VI is devoted to a summary and discussion.

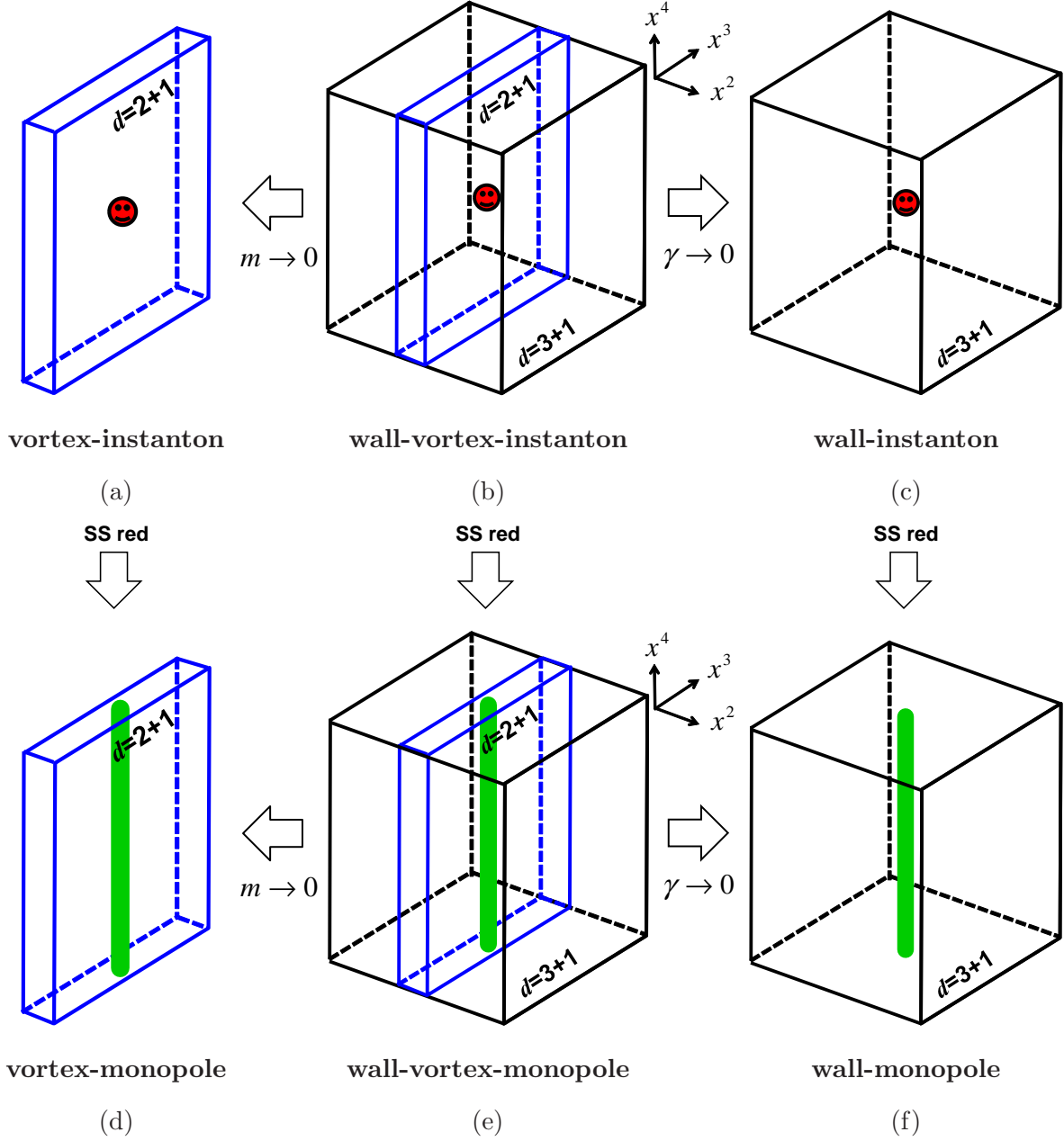


FIG. 1: Duality relations among all configurations. The codimensional direction  $x^1$  of the non-Abelian domain wall is not shown here. The (black) square boxes denote non-Abelian domain walls (Josephson junctions), (blue) thin boxes are vortices, (red) circles are Yang-Mills instantons, and (green) lines denote monopoles. (a) Instanton inside a non-Abelian vortex, (b) Instanton inside a non-Abelian vortex trapped in non-Abelian domain wall (c) Instanton inside a non-Abelian domain wall (Josephson junction), (d) Monopole inside a non-Abelian vortex, (e) Monopole inside a non-Abelian vortex trapped in non-Abelian domain wall (f) Monopole inside a non-Abelian domain wall (Josephson junction). (c), (d) and (e) are obtained from (a), (b) and (c), respectively, by the Scherk-Schwarz dimensional reduction or a T-duality.

## II. THE MODEL

The theory that we consider is a  $U(N)$  gauge theory in the Higgs phase in  $d = 3 + 1$  (or  $d = 4 + 1$ ) dimensions with the following matter contents: a  $U(N)$  gauge field  $A_\mu$ , two  $N$  by  $N$  charged complex scalar fields  $H = (H_1, H_2)$ , and a real adjoint  $N$  by  $N$  scalar field  $\Sigma(x)$ . The Lagrangian is given as follows:

$$\mathcal{L} = -\frac{1}{4g^2}\text{tr } F_{\mu\nu}F^{\mu\nu} + \frac{1}{g^2}\text{tr } (D_\mu\Sigma)^2 + \text{tr } |D_\mu H|^2 + \mathcal{L}_J - V \quad (1)$$

where  $V$  is the potential term

$$V = \frac{g^2}{4}\text{tr } (HH^\dagger - v^2\mathbf{1}_N)^2 + \text{tr } |\Sigma H - HM|^2, \quad (2)$$

and  $D_\mu$  is the covariant derivative, given by  $D_\mu H = \partial_\mu H - iA_\mu H$  and  $D_\mu\Sigma = \partial_\mu\Sigma - i[A_\mu, \Sigma]$ ,  $g$  is the gauge coupling constant that we take common for the  $U(1)$  and  $SU(N)$  factors of  $U(N)$ ,  $v$  is a real constant representing the vacuum expectation value of  $H$ , and  $M$  is a  $2N$  by  $2N$  mass matrix for  $H$  given below. Here,  $\mathcal{L}_J$  is a scalar coupling that we call the Josephson interaction

$$\mathcal{L}_J = -\gamma\text{tr } (H_1^\dagger H_2 + H_2^\dagger H_1) \quad (3)$$

motivated by the Josephson junction of two superconductors. Instead of this term, we may consider a quadratic Josephson term  $\mathcal{L}_{J,2} = -\gamma\text{tr } [(H_1^\dagger H_2)^2 + (H_2^\dagger H_1)^2]$  [31] that is a non-Abelian extension of the Josephson term in chiral p-wave superconductors [51]. Apart from the Josephson term  $\mathcal{L}_J$  (or  $\mathcal{L}_{J,2}$ ), the model is a truncation of the bosonic part of  $\mathcal{N} = 2$  supersymmetric theory (with eight supercharges) in  $d = 3 + 1$  (or  $4 + 1$ ) [25].

The  $U(N)$  gauge (color) symmetry acts on fields as

$$A_\mu \rightarrow gA_\mu g^{-1} + ig\partial_\mu g^{-1}, \quad H \rightarrow gH, \quad \Sigma \rightarrow g\Sigma g^{-1}, \quad g \in U(N)_\text{C}, \quad (4)$$

while the flavor (global) symmetry depends on the mass matrix; In the massless case  $M = 0$ , the flavor symmetry is the maximum  $SU(2N)$ . This is explicitly broken by the mass matrix  $M$  that we take

$$M = \text{diag.}(m\mathbf{1}_N, -m\mathbf{1}_N) \quad (5)$$

with a real constant  $m$ , together with a small mass perturbation

$$M = \text{diag.}(m\mathbf{1}_N + \Delta M, -m\mathbf{1}_N - \Delta M), \quad \Delta M = \text{diag.}(m_1, m_2, \dots, m_N) \quad (6)$$

where real mass shifts  $m_a$  are much smaller than  $m$ :  $m_a \ll m$ . For  $m \neq 0$  with  $\Delta M = 0$ , the flavor symmetry is  $SU(N)_L \times SU(N)_R \times U(1)_{L-R}$ , given by

$$H_1 \rightarrow H_1 U_L e^{+i\alpha}, \quad H_2 \rightarrow H_2 U_R e^{-i\alpha}, \quad U_{L,R} \in SU(N)_{L,R}, \quad e^{i\alpha} \in U(1)_{L-R}, \quad (7)$$

while for  $\Delta M \neq 0$  with non-degenerate mass perturbation  $m_a \neq m_b$  for  $a \neq b$ , the flavor symmetry is explicitly broken to  $U(1)_L^{N-1} \times U(1)_R^{N-1} \times U(1)_{L-R}$ :

$$H_1 \rightarrow H_1 U_L e^{+i\alpha}, \quad H_2 \rightarrow H_2 U_R e^{-i\alpha}, \quad U_{L,R} \in U(1)_{L,R}^{N-1} \subset SU(N)_{L,R}. \quad (8)$$

In this paper, we mostly consider this non-degenerate case.

The vacuum structures of the model are as follows. In the massless case  $m = 0$  and  $\Delta M = 0$ , the vacuum can be taken without the loss of generality as

$$H = (v\mathbf{1}_N, \mathbf{0}_N), \quad \Sigma = \mathbf{0}_N \quad (9)$$

by using the  $SU(2N)$  flavor symmetry. The unbroken symmetry is  $SU(N)_{C+L} \times SU(N)_R \times U(1)$ , in which the factor  $SU(N)_{C+L}$  is the color-flavor locked (global) symmetry. The moduli space of vacua is the complex Grassmann manifold [52]

$$Gr_{2N,N} \simeq \frac{SU(2N)}{SU(N) \times SU(N) \times U(1)}. \quad (10)$$

In the massive case,  $m \neq 0$  but still  $\Delta M = 0$ , the above vacua are split into the following two disjoint vacua [53]

$$\begin{aligned} H &= (v\mathbf{1}_N, \mathbf{0}_N), \quad \Sigma = +m\mathbf{1}_N : \quad SU(N)_{C+L}, \\ H &= (\mathbf{0}_N, v\mathbf{1}_N), \quad \Sigma = -m\mathbf{1}_N : \quad SU(N)_{C+R} \end{aligned} \quad (11)$$

with the unbroken color-flavor locked (global) symmetries  $g = U_L$  and  $g = U_R$ , respectively. These vacua are color-flavor locked vacua that can be interpreted as non-Abelian color superconductors.

With the non-degenerate mass deformation  $\Delta M \neq 0$ , each vacuum in Eq. (11) is shifted to

$$\begin{aligned} H &= (v'\mathbf{1}_N, \mathbf{0}_N), \quad \Sigma = +m\mathbf{1}_N + \Delta M : \quad U(1)_{C+L}^{N-1}, \\ H &= (\mathbf{0}_N, v'\mathbf{1}_N), \quad \Sigma = -m\mathbf{1}_N - \Delta M : \quad U(1)_{C+R}^{N-1}, \end{aligned} \quad (12)$$

where  $v'$  is shifted from  $v$ .

In the following sections, we often work in the strong coupling (nonlinear sigma model) limit  $g \rightarrow \infty$  for explicit calculations. In this limit, we have the constraints

$$HH^\dagger = v^2 \mathbf{1}_N, \quad (13)$$

$$\Sigma = \frac{HMH^\dagger}{HH^\dagger} = v^{-2}HMH^\dagger, \quad (14)$$

$$A_\mu = \frac{i}{2}v^{-2}[H\partial_\mu H^\dagger - (\partial_\mu H)H^\dagger], \quad (15)$$

and the model is reduced to the Grassmann sigma model with the target space given in Eq. (10) together with a potential term, known as the massive (twisted-mass deformed) Grassmann sigma model [54].

### III. NON-ABELIAN JOSEPHSON JUNCTION AND NON-ABELIAN JOSEPHSON VORTEX

#### A. Non-Abelian Josephson junction as a non-Abelian domain wall

For a while, we consider the case in the absence of the Josephson term  $\gamma = 0$  and the mass deformation  $\Delta M = 0$ , and then we turn them on later. In the sigma model limit, a non-Abelian domain wall solution interpolating between the two vacua in Eq. (11) can be obtained as [8–10, 17]

$$H_{\text{wall},0} = \frac{v}{\sqrt{1 + |u_{\text{wall}}|^2}} (\mathbf{1}_N, u_{\text{wall}} \mathbf{1}_N), \quad u_{\text{wall}}(x^1) = e^{\mp m(x^1 - X^1) + i\varphi}, \quad (16)$$

with  $\Sigma$  and  $A_1$  obtained from Eqs. (14) and (15), where we place it perpendicular to the  $x^1$  coordinate, and  $X^1$  is its position in that coordinate or the translational modulus.  $u_{\text{wall}}$  is a domain wall solution in the massive  $\mathbb{CP}^1$  model [15] with width  $m^{-1}$ . The most general solution can be obtained from the solution in Eq. (16) by acting the  $SU(N)_{\text{C+L+R}}$  symmetry that remains in the vacuum on the above solution:

$$H_{\text{wall}} = V H_{\text{wall},0} \begin{pmatrix} V^\dagger & 0 \\ 0 & V \end{pmatrix} = \frac{v}{\sqrt{1 + e^{\mp 2m(x^1 - X^1)}}} (\mathbf{1}_N, e^{\mp m(x^1 - X^1)} U), \quad (17)$$

with  $V \in SU(N)$ . Here we have defined the group-valued moduli  $U$  by  $U \equiv V^2 e^{i\varphi} \in U(N)$ . This transformation gives the domain wall the moduli  $U \in U(N)$  in addition to  $X^1$  [55]

$$(X^1, U) \in \mathcal{M}_{\text{wall}} \simeq \mathbb{R} \times U(N). \quad (18)$$



Now we turn on the Josephson interaction  $\gamma$  so that the domain wall becomes the Josephson junction. The effective theory of the non-Abelian Josephson junction can be constructed by using the (Manton's) moduli approximation [56, 57]; First, we promote the moduli parameters  $X^1$  and  $U$  to moduli fields  $X^1(x^i)$  and  $U(x^i)$ , respectively ( $i = 0, 2, 3, (4)$  for  $d = 3 + 1 (4 + 1)$ ) on the world volume of the domain wall, and then perform integration over the codimension. We thus obtain the effective theory given by [8–10]:

$$\mathcal{L}_{\text{wall}} = \frac{v^2}{2m} \partial_i X^1 \partial^i X^1 - f_\pi^2 \text{tr} (U^\dagger \partial_i U U^\dagger \partial^i U) + \mathcal{L}_{\text{wall},J}, \quad f_\pi^2 \equiv \frac{v^2}{4m} \quad (19)$$

with the mass term induced from the non-Abelian Josephson term [28]

$$\mathcal{L}_{\text{wall},J} = -m'^2 (\text{tr} U + \text{tr} U^\dagger), \quad m'^2 \equiv \frac{\pi\gamma}{2m}. \quad (20)$$

We thus obtain the  $U(N)$  chiral Lagrangian in Eq. (19), and the term in Eq. (20) is nothing but the conventional pion mass term. This potential term lifts the  $U(N)$  vacuum manifold, leaving the unique vacuum

$$U = \mathbf{1}_N, \quad (21)$$

as the case of the usual chiral Lagrangian.

## B. Non-Abelian vortex as a non-Abelian sine-Gordon soliton inside the junction

In this subsection, we discuss a non-Abelian vortex. First, we discuss a non-Abelian vortex in the absence of the Josephson junction, and later consider them together. The non-Abelian vortices in the massless case  $m = 0$  and  $\Delta M = 0$  are non-Abelian semi-local vortices [58], but in the massive case they become local vortices of the Abrikosov-Nielsen-Olesen (ANO) vortex type [59]. In the left (right) vacuum in Eq. (11), we can neglect  $H_2$  ( $H_1$ ). There, the  $U(N)$  symmetry is spontaneously broken completely, locking with the  $SU(N)_{L(R)}$  flavor symmetry to the  $SU(N)_{C+L(R)}$  color-flavor locked symmetry. A non-Abelian vortex solution in the  $x^1$ - $x^2$  plane with a non-Abelian magnetic field  $F_{12}$  and a scalar field, is given by

$$F_{12,0} = \text{diag}(F_*(r), 0, \dots, 0), \quad (H_{1(2)})_0 = v \text{diag}(f(r)e^{i\theta}, 1, \dots, 1), \quad (22)$$

with the boundary conditions for the profile function  $g$ ,  $g(r) \rightarrow 1$  ( $r \rightarrow \infty$ ) and  $g(r) \rightarrow 0$  ( $r = 0$ ). Here  $(r, \theta)$  are the polar coordinates in the  $x^1$ - $x^2$  plane. This solution is obtained by embedding of the ANO vortex solution [59]  $(F_*(r), g(r)e^{i\theta})$  into the upper-left

corner. The most general solution can be obtained by acting the color-flavor locked symmetry  $SU(N)_{C+L(R)}$  to the above solution:

$$F_{12} = V \text{diag}(F_*(r), 0, \dots, 0) V^\dagger, \quad H_{1(2)} = v V \text{diag}(f(r)e^{i\theta}, 1, \dots, 1) V^\dagger, \\ V \in SU(N). \quad (23)$$

This solution spontaneously breaks the color-flavor locked symmetry  $SU(N)_{C+L(R)}$  into a subgroup  $SU(N-1) \times U(1)$ . Therefore, it results in the moduli localized on the vortex core;

$$\mathcal{M}_{\text{vortex}} \simeq \mathbb{C} \times \mathbb{C}P^{N-1} = \mathbb{C} \times \frac{SU(N)_{C+L(R)}}{SU(N-1) \times U(1)}, \quad (24)$$

which are called the orientational moduli.

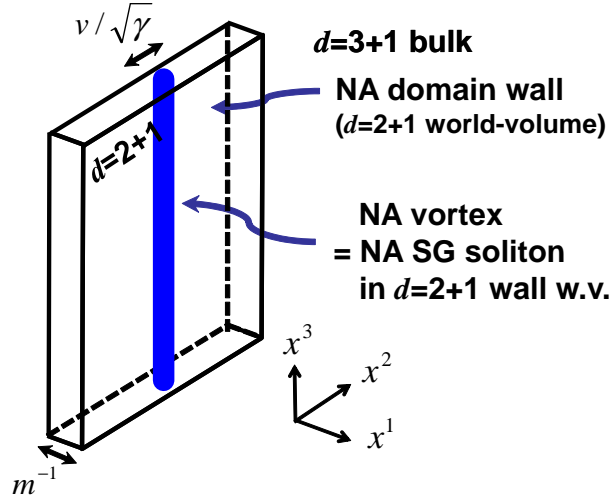


FIG. 2: A schematic picture of a non-Abelian sine-Gordon soliton in a non-Abelian domain wall describing a non-Abelian vortex.

When the non-Abelian vortex is placed parallel to the non-Abelian Josephson junction (domain wall), it is absorbed into the junction to minimize the total energy (see Fig. 2). The resulting configuration can be described as a non-Abelian sine-Gordon soliton in the  $U(N)$  chiral Lagrangian in Eq. (19) with the mass term in Eq. (20). A non-Abelian sine-Gordon soliton (perpendicular to the  $x^2$  coordinate) is given as [31, 39]:

$$U(x) = \text{diag}(u(x^2), 1, \dots, 1), \quad (25)$$

$$u(x^2) = \exp i\theta_{\text{SG}}(x^2) = \exp(4i \arctan \exp[m''(x^2 - X^2)]), \quad (26)$$

	$x^0$	$x^1$	$x^2$	$x^3$	$x^4$
NA wall (Josephson junc)	○	×	○	○	○
NA vortex	○	×	×	○	○
monopole	○	×	×	×	○
YM instanton	○	×	×	×	×

TABLE I: The space-time configurations of the topological solitons in this paper. Here, “ $\times$ ” denote the codimensions that soliton configurations depend on, while “ $\circ$ ” denote the world-volume directions that the static soliton configurations do not depend and the moduli fields live in.

with the translational modulus  $X^2$  and the effective mass  $m''$  defined by

$$m''^2 = \frac{m'^2}{f_\pi^2} = \frac{2\pi\gamma}{v^2}. \quad (27)$$

The width of the soliton is  $m''^{-1} \sim v/\sqrt{\gamma}$ , and the tension of the soliton is

$$T_{\text{SG}} = 8m''. \quad (28)$$

The most general single soliton solution can be obtained by acting the  $SU(N)$  symmetry on Eq. (26):

$$U(x) = V \text{diag}(u(x^2), 1, \dots, 1) V^\dagger, \quad V \in SU(N). \quad (29)$$

Therefore, the non-Abelian sine-Gordon soliton carries the orientational moduli

$$\mathcal{M}_{\text{NASG}} \simeq \mathbb{R} \times \mathbb{C}P^{N-1} \simeq \mathbb{R} \times \frac{SU(N)}{SU(N-1) \times U(1)}, \quad (30)$$

that coincides with the moduli of the non-Abelian vortex in Eq. (24), except for one translation modulus  $X^1$  transverse to the junction. The composite configuration (in the  $x^1$ - $x^2$  plane) can be written as

$$H_{\text{composite}} = \frac{1}{\sqrt{1 + e^{\mp 2m(x^1 - X^1)}}} \left( \mathbf{1}_N, e^{\mp m(x^1 - X^1)} V \text{diag}(e^{i\theta_{\text{SG}}(x^2)}, 1, \dots, 1) V^\dagger \right). \quad (31)$$

It was shown in Ref. [28] from the flux matching that this is precisely a non-Abelian vortex. The coordinates of the configurations were summarized in (the first two lines of) Table I. This composite configuration is non-BPS. In fact, the Josephson term stabilizing the vortex cannot be made supersymmetric.

The effective theory of the sine-Gordon soliton with the world-volume  $x^\alpha$  ( $\alpha = 0, 3, (4)$  for  $d = 3 + 1$  ( $4 + 1$ )) can be also obtained by the moduli approximation [39]:

$$\mathcal{L}_{\text{SG}} = C_X \partial_\alpha X^2 \partial^\alpha X^2 + C_\phi [\partial_\alpha \phi^\dagger \partial^\alpha \phi + (\phi^\dagger \partial_\alpha \phi)(\phi^\dagger \partial^\alpha \phi)] \quad (32)$$

with the constants (called Kähler classes)

$$C_X = \frac{f_\pi^2 T_{\text{SG}}}{2} = \sqrt{2\pi} \frac{v \sqrt{\gamma}}{m}, \quad C_\phi = \frac{f_\pi^2 T_{\text{SG}}}{m'^2} = \sqrt{\frac{2}{\pi}} \frac{v^3}{m \sqrt{\gamma}}. \quad (33)$$

Here, the first equalities were derived in Ref. [39] and the second equalities hold from Eqs. (27) and (28).

#### IV. INSTANTONS INSIDE A NON-ABELIAN JOSEPHSON JUNCTION: JOSEPHSON INSTANTONS

When we study Yang-Mills instantons, we promote the dimensionality of space-time to  $d = 4 + 1$ , and consider instanton-particles in four Euclidean space in  $d = 4 + 1$  dimensions.

First we consider an instanton with the help of a non-Abelian vortex far apart from the junction as illustrated in Fig. 3(a). The non-Abelian vortex has the moduli in Eq. (24). The effective theory of the non-Abelian vortex placed in the  $x^1$ - $x^2$  plane is therefore the  $\mathbb{CP}^{N-1}$  model ( $\alpha = 0, 3, 4$  for  $d = 4 + 1$ ) [6, 12, 20, 21]

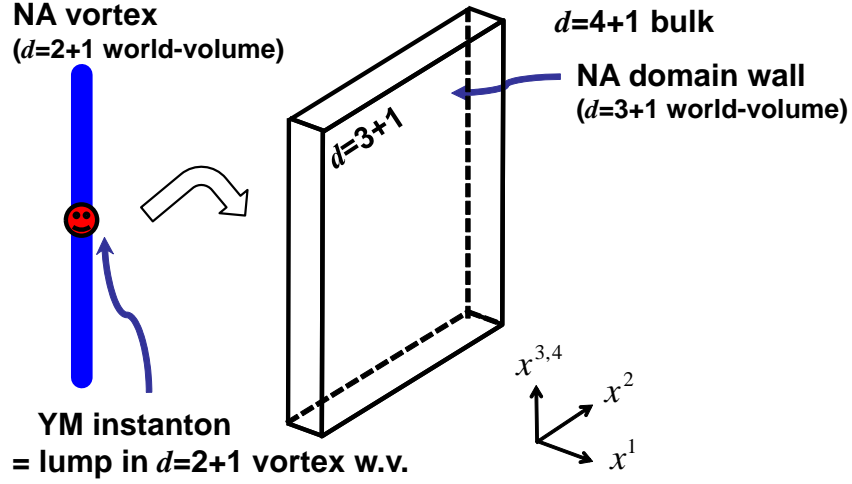
$$\mathcal{L}_{\text{vortex}} = 2\pi v^2 \partial_\alpha Z \partial^\alpha Z + \frac{4\pi}{g^2} [\partial_\alpha \phi^\dagger \partial^\alpha \phi + (\phi^\dagger \partial_\alpha \phi)(\phi^\dagger \partial^\alpha \phi)], \quad (34)$$

with the complex position moduli  $Z \equiv X^1 + iX^2$  of the vortex and a complex  $N$ -vector  $\phi$  with a constraint  $\phi^\dagger \phi = 1$ . The  $\mathbb{CP}^{N-1}$  model admits  $\mathbb{CP}^{N-1}$  lumps classified by  $\pi_2(\mathbb{CP}^{N-1}) \simeq \mathbb{Z}$ . The  $\mathbb{CP}^{N-1}$  lumps (in the  $x^3$ - $x^4$  plane) in the vortex effective theory can be identified with Yang-Mills instantons in the bulk [12]. This can be verified from the lump energy  $E_{\text{lump}}$ , coinciding with the instanton energy  $E_{\text{inst}}$  [12]:

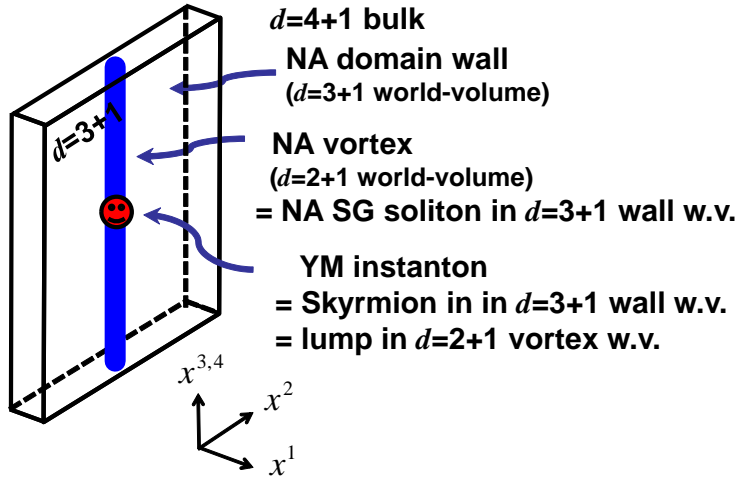
$$E_{\text{lump}} = \frac{4\pi}{g^2} T_{\text{lump}} = \frac{4\pi}{g^2} \times 2\pi k = \frac{8\pi^2}{g^2} k = E_{\text{inst}}. \quad (35)$$

Here,  $T_{\text{lump}} = 2\pi k$  is the lump charge with the lump number  $k \in \pi_2(\mathbb{CP}^{N-1}) \simeq \mathbb{Z}$ .

In the presence of the non-Abelian Josephson junction, this composite soliton, a vortex-instanton composite, will be absorbed into the junction to minimize the total energy. We then obtain an instanton inside the non-Abelian vortex inside the junction as illustrated



(a)



(b)

FIG. 3: A Yang-Mills instanton as a lump inside a non-Abelian vortex (a) apart from (b) inside a junction (domain wall).

in Fig. 3(b). The non-Abelian vortex in the bulk is a non-Abelian sine-Gordon soliton inside the junction whose effective theory is the  $U(N)$  principal chiral model. It was further shown in Ref. [39] that  $\mathbb{C}P^{N-1}$  lumps inside the non-Abelian sine-Gordon soliton are  $SU(N)$  Skymion in the  $U(N)$  principal chiral model as follows: The baryon (Skymion) number  $B$

taking a value in  $\pi_3[SU(N)] \simeq \mathbb{Z}$  in the bulk can be calculated as ( $i = 2, 3, 4$ )

$$\begin{aligned}
B &= \frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} \text{tr} (U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U) \\
&= -\frac{1}{8\pi^2} \int d^3x \text{tr} [(\partial_3 U^\dagger \partial_4 U - \partial_4 U^\dagger \partial_3 U) U^\dagger \partial_2 U] \\
&= \frac{i}{2\pi} \int dz d\bar{z} \text{tr} ([\partial_{\bar{z}} \mathcal{P}, \partial_z \mathcal{P}] \mathcal{P}) \times \frac{1}{2\pi} \int dx^2 (1 - \cos \theta_{\text{SG}}) \partial_2 \theta_{\text{SG}} \\
&= kl
\end{aligned} \tag{36}$$

with the projector  $\mathcal{P} = \phi \phi^\dagger$ ,  $z \equiv x^3 + ix^4$ , the lump number  $k \in \pi_2(\mathbb{CP}^{N-1}) \simeq \mathbb{Z}$ , and the sine-Gordon soliton number  $l \in \pi_1[U(1)] \simeq \mathbb{Z}$  defined by

$$l \equiv \frac{\theta_{\text{SG}}(x^2 = +\infty) - \theta_{\text{SG}}(x^2 = -\infty)}{2\pi}. \tag{37}$$

Therefore, the  $\mathbb{CP}^{N-1}$  lumps on the non-Abelian sine-Gordon soliton are  $SU(N)$  Skyrmions in the principal chiral model in  $d = 3 + 1$  dimensions. This composite configurations is non-BPS.

Finally, by following Ref. [9] we discuss that the  $SU(N)$  Skyrmions inside the non-Abelian Josephson junction (domain wall) can be identified with Yang-Mills instanton particles in the  $d = 4 + 1$  bulk. Far apart from the domain wall, the gauge field  $A_\mu$  falls into a pure gauge  $A_\mu = -i(\partial_\mu U)U^\dagger$ . In this setting, the instanton number coincides with the baryon number:

$$\begin{aligned}
I &= \frac{1}{24\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{1}{24\pi^2} \int_{\mathbb{R}^3(x^1=+\infty) - \mathbb{R}^3(x^1=-\infty)} d^3x \epsilon^{ijk} \text{tr} (U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U) \\
&= \int_{\mathbb{R}^3(x^1=+\infty)} d^3x \epsilon^{ijk} \text{tr} (U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U) = B.
\end{aligned} \tag{38}$$

Therefore, we have a consistent picture. The  $SU(N)$  Yang-Mills instantons are  $SU(N)$  Skyrmions in the non-Abelian domain wall (Josephson junction) and are  $\mathbb{CP}^{N-1}$  lumps inside the non-Abelian vortex as illustrated in Fig. 3(b).

We have two limits to remove host solitons as illustrated in Fig. 1 (a), (b) and (c), where we have drawn configurations inside the Josephson junction (wall), ignoring the outside. The two limits are: (1)  $m \rightarrow 0$ : the non-Abelian domain wall disappears in this limit, since its width is  $m^{-1}$ . The configuration is a vortex-instanton composite, where instantons are lumps in the vortex. This composite is 1/4 BPS if embedded into a supersymmetric theory [12].

bulk	NA vortex	NA wall (Josephson junc)
$SU(N)$ instanton	$\mathbb{C}P^{N-1}$ lump	$SU(N)$ Skymion
$\Downarrow$ SS red	$\Downarrow$ SS red	$\Downarrow$ SS red
$SU(N)/U(1)^{N-1}$ monopole	$\mathbb{C}P^{N-1}$ kink	$U(1)^{N-1}$ vortex

TABLE II: T-duality relations (Scherk-Schwartz dimensional reductions).  $SU(N)$  instantons are dimensionally reduced to  $SU(N)/U(1)^{N-1}$  (Abelian) monopoles in the bulk for non-degenerate twisted masses. This relation leads two duality relations;  $\mathbb{C}P^{N-1}$  lumps are dimensionally reduced to  $\mathbb{C}P^{N-1}$  kinks inside the non-Abelian vortex, while  $SU(N)$  Skyrmons are dimensionally reduced to the  $U(1)^{N-1}$  vortices inside the non-Abelian domain wall or Josephson junction.

(2)  $\gamma \rightarrow 0$ : the non-Abelian vortex disappears in this limit, since the size of the vortex along the domain wall world-volume is proportional to  $m''^{-1} \sim v/\sqrt{\gamma}$ . The configuration is a wall-instanton composite, where instantons are Skyrmons in the wall. This composite is non-BPS [26] even in the absence of the Josephson term which cannot be made supersymmetric.

Therefore, the original configuration gives a kind of duality between  $\mathbb{C}P^{N-1}$  lumps and  $SU(N)$  Skyrmons both realized by Yang-Mills instantons.

## V. MONOPOLES INSIDE A NON-ABELIAN JOSEPHSON JUNCTION: JOSEPHSON MONOPOLES

In this section, we discuss monopoles in the non-Abelian Josephson junction. We now turn on the mass perturbation  $\Delta M$  in Eq. (6). It is useful to note that this mass can be obtained from the theory with  $\Delta M = 0$  with the compactified  $x^4$  direction with the twisted boundary condition along the  $x^4$  coordinate:

$$H(x^\mu, x^4 + R) = H(x^\mu, x^4) \begin{pmatrix} \exp(i\Delta M) & \mathbf{0}_N \\ \mathbf{0}_N & \exp(-i\Delta M) \end{pmatrix} \quad (39)$$

with  $\mu = 0, 1, 2, 3$ . This twisting group element belongs to  $U(1)^{N-1}$  in the  $SU(N)$  flavor symmetry. By assuming the  $x^4$  dependence of the fields as

$$H(x^\mu, x^4) = H(x^\mu) \begin{pmatrix} \exp[i(x^4/R)\Delta M] & \mathbf{0}_N \\ \mathbf{0}_N & \exp[-i(x^4/R)\Delta M] \end{pmatrix}, \quad (40)$$

we substitute this to the kinetic term, to obtain the mass deformation  $\Delta M$  (we set  $R = 1$ ):

$$\partial_4 H(x^\mu, x^4) = i(H_1(x^\mu)\Delta M, -H_2(x^\mu)\Delta M). \quad (41)$$

With putting  $A_4(x^\mu, x^4) = \Sigma(x^\mu)$

$$\text{tr} |D_4 H(x^\mu, x^4)|^2 = \text{tr} |\Sigma H_1 - H_1(x^\mu)\Delta M|^2 + \text{tr} |\Sigma H_2 + H_2(x^\mu)\Delta M|^2. \quad (42)$$

This is known as the Scherk-Schwarz dimensional reduction. The Scherk-Schwarz dimensional reduction induces the twisted mass on the soliton world-volumes too.

### A. Monopoles inside a non-Abelian vortex

The Scherk-Schwarz dimensional reduction acts on the moduli fields on a non-Abelian vortex as

$$\phi(x^\alpha, x^4) = \exp[i(x^4/R)\Delta M]\phi(x^\alpha) \quad (43)$$

with the vortex world-volume coordinates  $x^\alpha$  ( $\alpha = 0, 3$ ). We then obtain

$$\begin{aligned} \mathcal{L}_{\text{vortex}, \Delta M} &= 2\pi v^2 \partial_\alpha Z \partial^\alpha Z + \frac{4\pi}{g^2} [\partial_\alpha \phi^\dagger \partial^\alpha \phi + (\phi^\dagger \partial_\alpha \phi)(\phi^\dagger \partial^\alpha \phi)] - V \\ V &= \frac{4\pi}{g^2} [(\phi^\dagger \Delta M \phi)^2 - \phi^\dagger (\Delta M)^2 \phi], \end{aligned} \quad (44)$$

that is known as the massive (or twisted-mass deformed)  $\mathbb{C}P^{N-1}$  model. For non-degenerate mass deformation  $\Delta M$ , this potential admits  $N$  discrete vacua

$$\phi_a^T = (0, \dots, 0, 1, 0, \dots), \quad a = 1, \dots, N \quad (45)$$

where only the  $a$ -component is nonzero. The Lagrangian (44) admits  $N - 1$  multi-kink solutions, where the constituent kink connecting the  $a$ -th and  $a + 1$ -th vacua has the mass  $E_{\text{kink}, a}$  ( $a = 1, \dots, N - 1$ ) that coincides with the mass of a monopole  $E_{\text{monopole}, a}$  [19–22]:

$$E_{\text{kink}, a} = \frac{4\pi}{g^2} (m_{a+1} - m_a) = E_{\text{monopole}, a}. \quad (46)$$

This coincidence implies the coincidence of topological charges, since they are both BPS. The monopole-vortex composite configuration is illustrated in Fig. 1(d).

In the presence of the Josephson junction, this monopole-vortex composite is absorbed into it, resulting in a wall-vortex-monopole composite as in Fig. 1(e), that is discussed in the following subsections.



## B. Monopoles inside a non-Abelian domain wall

Let us consider the case of  $\gamma = 0$  in this subsection, in which case we do not have any vortex. We work in the domain-wall effective theory. The Scherk-Schwarz dimensional reduction along the compactified direction  $x^4$  acts on the moduli fields  $U$  on a non-Abelian domain wall as

$$U(x^i, x^4) = \exp[i(x^4/R)\Delta M]U(x^i)\exp[-i(x^4/R)\Delta M] \quad (47)$$

with  $i = 0, 2, 3$ . Then, the derivative of the moduli with respect to the compactified direction is obtained as

$$\partial_4 U(x^i, x^4) = i[\Delta M, U(x^i, x^4)], \quad (48)$$

and the gradient term in the  $x^4$  direction can be calculated as

$$\text{tr}(iU^\dagger(x^i, x^4)\partial_4 U(x^i, x^4))^2 = -\text{tr}([\Delta M, U(x^i)]^\dagger[\Delta M, U(x^i)]). \quad (49)$$

We thus obtain the domain-wall effective theory (or the effective theory of the Josephson junction) in the presence of the twisted mass  $\Delta M$  in the original theory:

$$\begin{aligned} \mathcal{L}_{\text{wall}, \Delta M} &= \frac{v^2}{2m} \partial_i X \partial^i X - \frac{v^2}{4m} \text{tr}(U^\dagger \partial_i U U^\dagger \partial^i U) - V \\ V &= \frac{v^2}{4m} \text{tr}([\Delta M, U]^\dagger[\Delta M, U]) \end{aligned} \quad (50)$$

The vacua of the domain-wall effective theory are given by the condition

$$[\Delta M, U] = 0. \quad (51)$$

When  $\Delta M$  is non-degenerate, Eq. (6) with  $m_a \neq m_b$  for  $a \neq b$ , the moduli space  $M$  of vacua is

$$U = \text{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_N}) : \quad M \simeq U(1)^{N-1} \quad (52)$$

with  $\sum_{a=1}^N \alpha_a = 0$ . It has the nontrivial first homotopy group

$$\pi_1(M) \simeq \mathbb{Z}^{N-1} \quad (53)$$

admitting  $N-1$  kinds of vortices. These  $N-1$  kinds of vortices correspond to the monopole charge  $\pi_2[SU(N)/U(1)^{N-1}] \simeq \pi_1[U(1)^{N-1}] \simeq \mathbb{Z}^{N-1}$  and to  $N-1$  kinks in the mass deformed  $\mathbb{C}P^{N-1}$  model.

To be more specific, let us consider the simplest case of  $N = 2$ . In this case, the target space of the principal chiral model is  $SU(2) \simeq S^3 \simeq O(4)/O(3)$  (except for the  $U(1)$  part), that is the  $O(4)$  model. Let us express the field  $U$  in terms of four real scalar fields  $n_A(x)$  ( $A = 1, 2, 3, 4$ ) with the constraint  $\sum_A n_A^2 = 1$ :

$$U = i \sum_{a=1}^3 n_a \sigma^a + n_4 \mathbf{1}_2, \quad (54)$$

where  $\sigma^a$  are the Pauli matrices and  $U^\dagger U = \mathbf{1}_2$  is equivalent to  $\mathbf{n} \cdot \mathbf{n} = 1$ . The Lagrangian is

$$\mathcal{L} = \frac{v^2}{4m} \partial_i \mathbf{n} \cdot \partial^i \mathbf{n} \quad (55)$$

The twisted boundary condition

$$U(x^i, x^4) = e^{i(x^4/R)m_1\sigma_3} U(x^i) e^{-i(x^4/R)m_1\sigma_3} \quad (56)$$

can be rewritten as

$$(n_1, n_2, n_3, n_4)(x^i, x^4) = \left( \hat{n}_1(x^i) \cos \frac{m_1}{R} x^4, \hat{n}_2(x^i) \sin \frac{m_1}{R} x^4, \hat{n}_3(x^i), \hat{n}_4(x^i) \right), \quad (57)$$

and the induced potential term is (we take  $R = 1$ )

$$V_m = \frac{v^2}{4m} \int_0^R dx^4 [(\partial_4 n_1)^2 + (\partial_4 n_2)^2] = \frac{v^2 m_1^2}{16m} (\hat{n}_1^2 + \hat{n}_2^2) = \frac{v^2 m_1^2}{16m} (1 - \hat{n}_3^2 - \hat{n}_4^2). \quad (58)$$

Numerical solutions in the case of  $N = 2$  were constructed before in Ref. [38].

The vacuum condition  $n_1 = n_2 = 0$  gives the vacuum manifold  $M \simeq S^1$ :  $n_3^2 + n_4^2 = 1$ . Therefore, the first homotopy group  $\pi_1(M) \simeq \mathbb{Z}$  admits one kind of a vortex. A vortex solution is of the form:

$$n_3 + in_4 = \cos f(r) e^{i\theta}, \quad n_1 + in_2 = \sin f(r) e^{i\alpha}, \quad (59)$$

$$U = \begin{pmatrix} \cos f(r) e^{i\theta} & -\sin f(r) e^{-i\alpha} \\ \sin f(r) e^{i\alpha} & \cos f(r) e^{-i\theta} \end{pmatrix}, \quad (60)$$

where  $f$  is a profile function satisfying the boundary conditions

$$f \rightarrow \pi/2 \text{ for } r \rightarrow \infty, \quad f = 0 \text{ for } r = 0, \quad (61)$$

where  $(r, \theta)$  are polar coordinates in the  $x^2$ - $x^3$  plane. Here,  $\alpha$  in Eq. (59) is a real constant representing a  $U(1)$  modulus of the vortex. This vortex inside the non-Abelian domain wall

(Josephson junction) is a monopole in the bulk, see Fig. 1(f). In fact, they have the same  $U(1)$  moduli.

For general  $N$ , a vortex solution corresponding to the  $a$ -th monopole ( $a$ -th kink) can be obtained by embedding the  $N = 2$  solution in Eq. (60) to  $N$  by  $N$  matrix  $U$  as

$$U_a = \begin{pmatrix} \mathbf{1}_{a-1} & & \\ & \cos f_a(r)e^{i\theta} & -\sin f_a(r)e^{-i\alpha_a} \\ & \sin f_a(r)e^{i\alpha_a} & \cos f_a(r)e^{-i\theta} \\ & & & \mathbf{1}_{N-a-1} \end{pmatrix} \quad (a = 1, \dots, N-1). \quad (62)$$

Here, the profile function  $f_a$  with the same boundary condition with Eq. (61) should depends on the masses  $m_a$  and  $m_{a+1}$ . The real constant  $\alpha_a$  is the  $U(1)$  modulus of the  $a$ -th vortex corresponding to that of the  $a$ -th monopole.

This composite is non-BPS [12] even in the absence of the Josephson term.

### C. Monopoles inside a Josephson vortex

The effect of  $\Delta M$  on the effective theory of a non-Abelian sine-Gordon soliton, that corresponds to Josephson vortex, *i.e.*, a vortex inside the non-Abelian domain wall can be obtained by the Scherk-Schwarz dimensional reduction as before. The Scherk-Schwarz dimensional reduction acts on the moduli  $\phi$  of the sine-Gordon soliton in the exactly same manner with those of the non-Abelian vortex in Eq. (43). We then obtain the effective theory of the non-Abelian sine-Gordon soliton, or the Josephson vortex, given by

$$\begin{aligned} \mathcal{L}_{\text{SG}} &= C_X \partial_\alpha X \partial^\alpha X + C_\phi [\partial_\alpha \phi^\dagger \partial^\alpha \phi + (\phi^\dagger \partial_\alpha \phi)(\phi^\dagger \partial^\alpha \phi)] - V \\ V &= C_\phi [(\phi^\dagger \Delta M \phi)^2 - \phi^\dagger (\Delta M)^2 \phi]. \end{aligned} \quad (63)$$

This is again the massive  $\mathbb{CP}^{N-1}$  model admitting a  $\mathbb{CP}^{N-1}$  kink that represents a monopole in the bulk. We then obtain the configuration in Fig. 1(e).

A nontrivial consistency check can be done from the domain-wall effective theory. Let us turn on  $\gamma \neq 0$  in the domain-wall effective theory in Eq. (50):

$$\begin{aligned} \mathcal{L}_{\text{wall}, \Delta M} &= \frac{v^2}{2m} \partial_i X \partial^i X - \frac{v^2}{4m} \text{tr} (U^\dagger \partial_i U U^\dagger \partial^i U) - V \\ V &= \frac{v^2}{4m} \text{tr} ([\Delta M, U]^\dagger [\Delta M, U]) + m'^2 (\text{tr} U + \text{tr} U^\dagger) \end{aligned} \quad (64)$$

and consider its effect on the vortex.

For the  $N = 2$  case, the domain-wall effective theory is

$$\begin{aligned}\mathcal{L}_{\text{wall}, \Delta M} &= \frac{v^2}{4m} \partial_i \mathbf{n} \cdot \partial^i \mathbf{n} - V \\ V &= \frac{v^2 m_1^2}{16m} (n_1^2 + n_2^2) + 2m'^2 n_4.\end{aligned}\tag{65}$$

In this case, the vortex in Eq. (59) or (60) is attached by two sine-Gordon solitons with the correct tension  $T = 8m''$  in Eq. (28). This can be manifest at large distance from the vortex core in Eq. (60). In the absence of  $\gamma$ , the field asymptotically goes to

$$U \rightarrow \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ for } r \rightarrow \infty.\tag{66}$$

In the presence of  $\gamma$ , this  $\theta$  dependence should be replaced by two sine-Gordon solitons

$$\begin{aligned}U &\rightarrow \begin{pmatrix} e^{i\theta_{\text{SG}}(x^2)} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \exp(4i \arctan \exp[m'(x^2 - X)]) & 0 \\ 0 & 1 \end{pmatrix} \text{ for } x^1 \rightarrow +\infty, \\ U &\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_{\text{SG}}(x^2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \exp(4i \arctan \exp[m'(x^2 - X)]) \end{pmatrix} \text{ for } x^1 \rightarrow -\infty, \\ U &\rightarrow \mathbf{1}_2 \text{ for } x^2 \rightarrow \pm\infty,\end{aligned}\tag{67}$$

where we have used Eq. (26). This deformation is illustrated in Fig. 4. Note that the direction of the path  $D_2$  at  $x^1 \rightarrow -\infty$  in Eq. (67) is opposite to the left side  $C_2$  of the angular path  $C_1 + C_2$  in Eq. (66) so that the lower-right components of  $U$ 's in Eqs. (66) and (67) have the same windings. These sine-Gordon solitons carry opposite  $\mathbb{CP}^1$  moduli. The sine-Gordon solitons are vortices from the bulk point of view. Therefore, we have shown that the monopole must be confined by the two vortices with the opposite  $\mathbb{CP}^1$  moduli. We then again reach the configuration in Fig. 1(e).

For general  $N$ , the asymptotic form of the  $a$ -th vortex becomes

$$U_a \rightarrow \begin{pmatrix} \mathbf{1}_{a-1} & & \\ & e^{i\theta} & 0 \\ & 0 & e^{-i\theta} \\ & & & \mathbf{1}_{N-a-1} \end{pmatrix} \quad (a = 1, \dots, N-1) \text{ for } r \rightarrow \infty\tag{68}$$

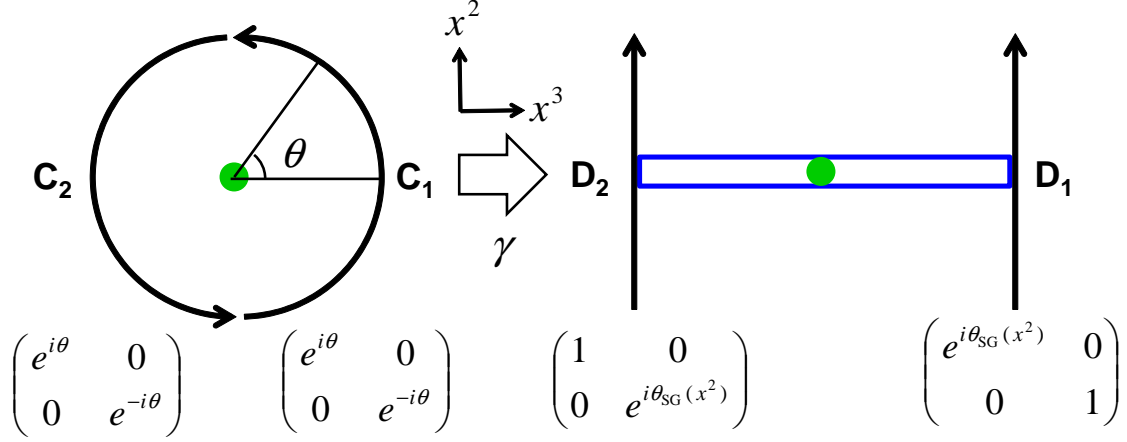


FIG. 4: Deformation of a unconfined monopole with  $\gamma = 0$  to a monopole confined by vortices with  $\gamma \neq 0$ , corresponding to Fig. 1 (f) and (e), respectively. Monopoles are vortices in the  $SU(2)$  principal chiral model with the mass deformation  $\Delta M$  realized inside the non-Abelian domain wall (see Fig. 1). The paths enclosing the vortices are the circular path  $C_1 + C_2$  for the unconfined monopole (left panel) and  $D_1 - D_2$  (plus the two paths at  $x^2 = \pm\infty$  where  $U$  is constant:  $U = \mathbf{1}_2$ ) for the confined monopole (right panel).

for  $\gamma = 0$ . In the presence of  $\gamma$ , this  $\theta$  dependence should be replaced by two sine-Gordon solitons as before:

$$\begin{aligned}
 U_a &\rightarrow \begin{pmatrix} \mathbf{1}_{a-1} & & \\ & e^{i\theta_{SG}(x^2)} & 0 \\ & 0 & 1 \\ & & & \mathbf{1}_{N-a-1} \end{pmatrix} \text{ for } x^1 \rightarrow +\infty, \\
 U_a &\rightarrow \begin{pmatrix} \mathbf{1}_{a-1} & & \\ & 1 & 0 \\ & 0 & e^{i\theta_{SG}(x^2)} \\ & & & \mathbf{1}_{N-a-1} \end{pmatrix} \text{ for } x^1 \rightarrow -\infty, \\
 U &\rightarrow \mathbf{1}_N \text{ for } x^2 \rightarrow \pm\infty.
 \end{aligned} \tag{69}$$

For composite solitons, the first and second paths are  $D_a$  and  $D_{a+1}$ , respectively, in Fig. 5.

The wall-vortex-monopole composites studied here are non-BPS. In general, wall-vortex-monopole composites can be 1/4 BPS if embedded into a supersymmetric theory, only when the vortices are perpendicular to the domain wall [18, 26].

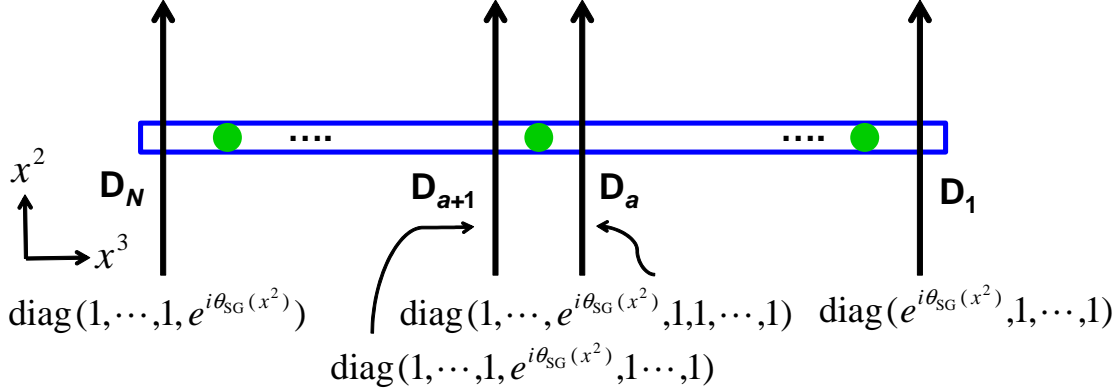


FIG. 5: Confined monopoles are vortices in the  $SU(N)$  principal chiral model with the mass deformation  $\Delta M$  realized inside the non-Abelian domain wall (see Fig. 1). The path enclosing the  $a$ -th vortex is  $D_a - D_{a+1}$  (plus the two paths at  $x^2 = \pm\infty$  where  $U$  is constant:  $U = \mathbf{1}_N$ ).

As for instantons, we have two limits to remove host solitons.

- (1)  $m \rightarrow 0$ : the non-Abelian domain wall disappears in this limit, since the width of it is  $m^{-1}$ . The configuration is a vortex-monopole composite, where monopoles are  $\mathbb{C}P^{N-1}$  kinks in the vortex. This composite is 1/4 BPS if embedded into a supersymmetric theory [12, 26].
- (2)  $\gamma \rightarrow 0$ : the non-Abelian vortex disappears in this limit, since the size of the vortex along the domain wall world-volume is proportional to  $m''^{-1} \sim v/\sqrt{\gamma}$ . The configuration is a wall-monopole composite, where monopoles are vortices in the wall. This composite is non-BPS [26]. Therefore, the original configuration gives a duality between  $\mathbb{C}P^{N-1}$  kinks and  $U(1)^{N-1}$  vortices both realized by Yang-Mills instantons.

## VI. SUMMARY AND DISCUSSION

We have studied instantons and monopoles in a non-Abelian Josephson junction, that is, a junction of non-Abelian color superconductors sandwiching an insulator. Low-energy dynamics of a non-Abelian domain wall can be described by a  $U(N)$  principal chiral model, where a non-Abelian Josephson vortex, a non-Abelian vortex (color magnetic flux tube) residing inside the junction, is described as a non-Abelian sine-Gordon soliton. Josephson instantons and monopoles have been realized inside the non-Abelian Josephson vortex in-

side the junction. By removing the junction with the vanishing Higgs mass,  $m = 0$ , the configurations go back to the well-known instanton-vortex and monopole-vortex composites. On the other hand, if we remove the vortex by turning off the Josephson coupling  $\gamma$ , there remain (unconfined) instantons and monopoles inside the junction (instanton-wall and monopole-wall composites). The whole situation is illustrated in Fig. 1. We have found that monopoles become  $U(1)^{N-1}$  vortices in the  $U(N)$  principal chiral model inside the junction, with the matching of the monopole charge  $\pi_2[SU(N)/U(1)^{N-1}] \simeq \mathbb{Z}^{N-1}$  and the vortex charge  $\pi_1[U(1)^{N-1}] \simeq \mathbb{Z}^{N-1}$ , while it was known that instantons become Skyrmons there. We have confirmed the monopole confinement in the junction; when we turn on the Josephson coupling  $\gamma$ , the  $U(1)^{N-1}$  vortices must be confined by the non-Abelian sine-Gordon solitons, implying that the monopole must be confined by the non-Abelian vortices in the bulk point of view. As summarized in Table II, we have shown that the T-duality relations between instantons and monopoles induces the duality between the  $SU(N)$  Skyrmons and the  $U(1)^{N-1}$  vortices (inside the junction) as well as the previously-known duality between  $\mathbb{C}P^{N-1}$  lumps and  $\mathbb{C}P^{N-1}$  kinks (inside the vortex). We have also observed the new kind of duality between  $\mathbb{C}P^{N-1}$  lumps and  $SU(N)$  Skyrmons and that between  $\mathbb{C}P^{N-1}$  kinks and  $U(1)^{N-1}$  vortices, as well as that between fractional instantons and bions in the  $\mathbb{C}P^{N-1}$  model in two Euclidean dimensions and those in the  $SU(N)$  principal chiral model in three Euclidean dimensions.

When we add the Chern-Simmons term in the gauge theory in the 4+1 dimensional bulk, the Wess-Zumino-Witten term is induced on the domain wall world-volume. This term would be important to interpret the domain wall world-volume theory as the low-energy effective theory of QCD.

In this paper, we have considered  $U(N)$  gauge theory but  $SU(N)$  gauge group does not change the main results, implying that those can be applied to color superconductors appearing in high density quark matter [29, 30], where non-Abelian vortices are superfluid vortices with color magnetic fluxes confined inside their cores [60]. If quark matter is separated by an insulator for instance by some modulation such as crystalline superconductivity, it will give (an array of) non-Abelian Josephson junctions. Non-Abelian vortices, monopoles and instantons there become non-Abelian Josephson vortices, Josephson monopoles and Josephson instantons, respectively, by trapped inside the insulating region.

As mentioned in introduction our configurations suggest a duality between fractional

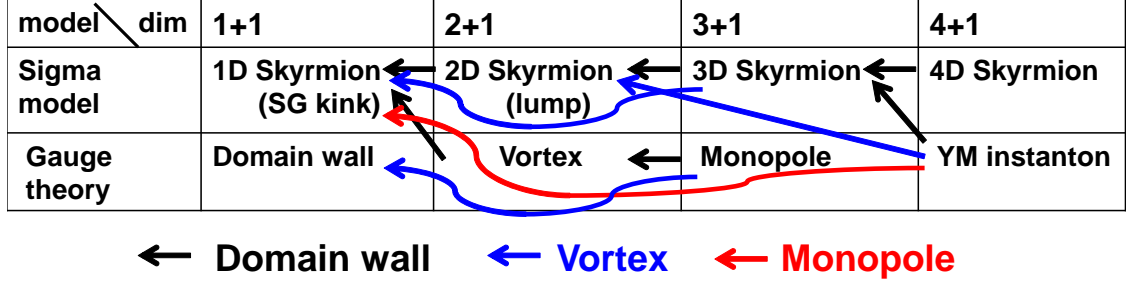


FIG. 6: Unification of topological solitons and instantons. The starting points of arrows are solitons in the bulk which become the endpoints of arrows when reside inside host solitons. The length of the arrows denote the host solitons: The black, blue and red arrows connecting one, two and three columns denote a domain wall, vortex and monopole.

instantons and bions in the  $\mathbb{CP}^{N-1}$  model on  $\mathbb{R}^1 \times S^1$  and the  $SU(N)$  principal chiral model on  $\mathbb{R}^2 \times S^1$  with twisted boundary conditions. Hopefully this duality may be useful to understand the resurgence of these models together with  $SU(N)$  Yang-Mills theory on  $\mathbb{R}^3 \times S^1$  at quantum level from a unified point of view.

Let us mention what the results in this paper imply for unified understanding of topological solitons and instantons. Various relations between host and daughter solitons found thus far are summarized in Fig. 6. Our new finding here is that a monopole becomes a vortex inside a non-Abelian domain wall.

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